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Arbitrary spin field equations on curved manifolds with torsion

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Abstract. In previous work a particularly simple set of arbitrary spin field equations for fields propagating on curved manifolds without torsion was proposed and studied. The consistency of those equations put some very strong conditions on the kind of background gravitational fields allowed, given that a certain quantum field did not vanish. As a prelude to an extensive study of index theorems and anomalies in the case of non-zero torsion, new arbitrary spin field equations are developed and the consistency conditions discussed. Some of the machinery needed for future investigations of arbitrary spin fields is developed using the two-component spinor formalism.

1. Introduction

Standard general relativity theory can be extended in many ways. One popular way is through the addition of torsion. On the classical level a great deal of work has been done on this possibility. Until recently very little had been done at the quantum level. Motivated by the successes and limitations of supergravity, which is a torsion theory, this is changing. Numerous papers are now appearing that treat some aspect of field theory on curved manifolds with torsion.

In this paper we begin a generalisation of some earlier work (Christensen and Duff 1979) to the case of non-zero torsion. In that paper we covered many topics: conformal and axial anomalies, index theorems and super index theorems. The analysis of all of these was based on a set of arbitrary spin field equations on curved manifolds. The equations were found to have many interesting properties. In particular, we found that in order for certain fields to exist on a manifold, restrictions on the curvature called consistency conditions were required. These conditions are very severe for fields with spins higher than two. In fact, based on the field equations we presented, there seemed to be no easy way to have physical fields with spins higher than two on anything but a flat manifold. Since we were mainly interested in applying our results to low spin theories like supergravity, we were not too concerned with this problem.

Interest in spin-$\frac{5}{2}$ 'hypergravity' theory has increased because of the limitations of supergravity. It has been proposed that hypertheories could have a bigger particle spectrum corresponding more closely with the actual elementary particle spectrum. We shall not pursue this line of thought here. Rather, we wish only to see how the addition of torsion changes the field equations and their consistency conditions. This paper is designed to present a foundation for later work. There is considerable detail involved in some of the sections. It has been pointed out (Christensen 1980) that torsion theories can be extremely complicated. Nothing we present here will contradict that opinion.

The amount of literature on torsion, higher spin field equations, supergravity and hypergravity is enormous. We will list a representative collection of papers. These papers and the references in them give a good overview of the topics covered in our work.

For a review of classical torsion theory see Hehl *et al* (1976). Supergravity theory is summarised nicely in van Nieuwenhuizen (1981). Discussions of spin-$\frac{5}{2}$ hypergravity can be found in Aragone (1981), Aragone and Deser (1979a, b, 1980), Berends *et al* (1979a, b, 1980), and van Holten (1979). Information on higher spin fields in general is found in Buchdahl (1958, 1962), Christensen and Duff (1979), Curtright (1979), Dowker and Dowker (1966), Fang and Fronsdal (1978), Freedman (1979), Fronsdal (1978, 1979), Lichnerowicz (1961, 1964), Penrose (1968) and Singh and Hagen (1974a, b). Recent work focusing on low spin fields propagating on curved manifolds with torsion, as well as the conformal and axial anomalies and index theorems associated with them, has been presented in Barvinsky and Vilkovisky (1981), Christensen (1980), Goldthorpe (1980), Kimura (1981), Nieh and Yan (1982) and Obukhov (1982).

In § 2 we introduce the properties of the torsion and Riemann tensor. We note that the symmetries of the Riemann tensor and Ricci tensor change when torsion is added. Useful tensor and two-component spinor notations are developed. We have found that two-component spinors provide the best formalism for studying arbitrary spin field equations and their properties.

In our previous work we found that two-component spinor decompositions of tensors were the easiest quantities to use when considering the consistency conditions imposed on the curvature. Decompositions of this type are presented in § 3. As we might expect, they are somewhat more complicated in the non-zero torsion case.

Ultimately we will want to compare our torsion results with the torsion-free cases. Section 4 provides the tensor and spinor relations needed to do this.

The two-component spinor Ricci identity derived in Pirani (1964) is a key element in the study of higher spin field equations and their consistency conditions. In § 5 we use some of the efforts of the previous sections to obtain the Ricci identities with torsion.

In § 6 we present our arbitrary spin field equations. These are the easiest generalisation we could find of our torsion-free equations. We do not pretend that these are the only set of equations possible. They may not be the correct ones. Too little is known at this time about the physical requirements. We do, however, believe that they provide a very good specimen of the kind of equations that will undoubtedly appear.

It is important to point out a few things about our field equations. First, the equations are really field equations for fields transforming according to the (A, B) representation of the group SO(4). Physical fields are in general some combination of these fields determined by requirements of gauge invariance. We do not examine questions of gauge invariance of our equations or whether they can be derived from an action principle. It is not always possible, as far as we see it, to demand gauge invariance. Gauge invariance is often used to ensure that physical particles propagate without tachyons or ghosts. However, there are examples of theories that have gauge invariant actions but contain particles with tachyonic or ghost properties. Fourth-order gravity theory is such a theory. The unphysical particles have to be treated in some other way.

We can, of course, require that each field have gauge invariant field equations. This restriction, along with the consistency conditions, can force the torsion to be zero or compel us to modify the couplings of the fields to the torsion. We plan to consider this in some future publication.

The most important result of § 6 is that the consistency conditions are *not* necessarily as restrictive as the torsion-free conditions of earlier work.

2. Notation, conventions and definitions

Picking a good set of conventions is sometimes more difficult than the problem being considered. We pick our notations and definitions so that they are as consistent with Misner *et al* (1973), Pirani (1964), Christensen and Duff (1979) and Christensen (1980) as we can make them. The choices made have served us well in performing the complicated computations involved in this work.

We consider quantum fields propagating on curved manifolds. For simplicity the manifolds will be compact without boundary. The metric signature will be $(+, +, +, +)$, that is, we work in a Euclidean regime. We will work with two different connections on the manifold, one symmetric connection (the Christoffel symbol)

$$\Gamma^{\alpha}{}_{\beta\gamma} \equiv \tfrac{1}{2} g^{\alpha\delta}(\partial_{\gamma} g_{\beta\delta} + \partial_{\beta} g_{\gamma\delta} - \partial_{\delta} g_{\beta\gamma}) \tag{2.1}$$

(Greek indices take values 0, 1, 2, 3), and one non-symmetric connection $\tilde{\Gamma}^{\alpha}{}_{\beta\gamma}$, to be defined by demanding that it be 'metric compatible'. If the covariant derivative $\tilde{\nabla}_{\mu}$ is defined using $\tilde{\Gamma}^{\alpha}{}_{\beta\gamma}$, metric compatibility is

$$\tilde{\nabla}_{\mu} g_{\alpha\beta} = \partial_{\mu} g_{\alpha\beta} - \tilde{\Gamma}^{\rho}{}_{\alpha\mu} g_{\rho\beta} - \tilde{\Gamma}^{\rho}{}_{\beta\mu} g_{\alpha\rho} = 0, \tag{2.2}$$

where ∂_{μ} is the usual partial derivative with respect to x^{μ}. Using (2.1), it is easy to see that

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + \tfrac{1}{2} T_{\beta\gamma}{}^{\alpha} - \tfrac{1}{2} T^{\alpha}{}_{\beta\gamma} - \tfrac{1}{2} T^{\alpha}{}_{\gamma\beta} \tag{2.3}$$

where

$$T_{\beta\gamma}{}^{\alpha} \equiv \tilde{\Gamma}^{\alpha}{}_{\beta\gamma} - \tilde{\Gamma}^{\alpha}{}_{\gamma\beta} \tag{2.4}$$

is the torsion tensor with

$$T_{\gamma\beta}{}^{\alpha} = -T_{\beta\gamma}{}^{\alpha}. \tag{2.5}$$

We define the contorsion tensor via

$$K_{\beta\gamma}{}^{\alpha} = \tfrac{1}{2}(T_{\beta\gamma}{}^{\alpha} - T^{\alpha}{}_{\beta\gamma} - T^{\alpha}{}_{\gamma\beta}), \tag{2.6}$$

so that

$$\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + K_{\beta\gamma}{}^{\alpha}. \tag{2.7}$$

Lowering the α index, we note that

$$K_{\alpha\gamma\beta} = -K_{\beta\gamma\alpha}. \tag{2.8}$$

The Riemann tensor for connection $\tilde{\Gamma}^{\alpha}{}_{\beta\gamma}$ is

$$\tilde{R}^{\alpha}{}_{\beta\gamma\delta} = \partial_{\gamma}\tilde{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_{\delta}\tilde{\Gamma}^{\alpha}{}_{\beta\gamma} + \tilde{\Gamma}^{\varepsilon}{}_{\beta\delta}\tilde{\Gamma}^{\alpha}{}_{\varepsilon\gamma} - \tilde{\Gamma}^{\varepsilon}{}_{\beta\gamma}\tilde{\Gamma}^{\alpha}{}_{\varepsilon\delta} \tag{2.9}$$

and

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\varepsilon{}_{\beta\delta}\Gamma^\alpha{}_{\varepsilon\gamma} - \Gamma^\varepsilon{}_{\beta\gamma}\Gamma^\alpha{}_{\varepsilon\delta} \tag{2.10}$$

for $\Gamma^\alpha{}_{\beta\gamma}$. The symmetries of $\tilde{R}_{\alpha\beta\gamma\delta}$ are

$$\tilde{R}_{\beta\alpha\gamma\delta} = -\tilde{R}_{\alpha\beta\gamma\delta}, \qquad \tilde{R}_{\alpha\beta\delta\gamma} = -\tilde{R}_{\alpha\beta\gamma\delta} \tag{2.11}$$

while for $R_{\alpha\beta\gamma\delta}$ they are

$$R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}, \qquad R_{\alpha\beta\delta\gamma} = -R_{\alpha\beta\gamma\delta}, \tag{2.12}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}. \tag{2.13}$$

The cyclic identities are

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0, \tag{2.14}$$

$$\tilde{R}_{\alpha\beta\gamma\delta} + \tilde{R}_{\alpha\gamma\delta\beta} + \tilde{R}_{\alpha\delta\beta\gamma} = -\tilde{\nabla}_\delta T_{\beta\gamma\alpha} - \tilde{\nabla}_\beta T_{\gamma\delta\alpha} - \tilde{\nabla}_\gamma T_{\delta\beta\alpha} + T_{\beta\gamma}{}^\varepsilon T_{\varepsilon\delta\alpha} + T_{\gamma\delta}{}^\varepsilon T_{\varepsilon\beta\alpha} + T_{\delta\beta}{}^\varepsilon T_{\varepsilon\gamma\alpha}, \tag{2.15}$$

and the Bianchi identities are

$$\nabla_\varepsilon R_{\alpha\beta\gamma\delta} + \nabla_\gamma R_{\alpha\beta\delta\varepsilon} + \nabla_\delta R_{\alpha\beta\varepsilon\gamma} = 0, \tag{2.16}$$

$$\tilde{\nabla}_\varepsilon \tilde{R}_{\alpha\beta\gamma\delta} + \tilde{\nabla}_\gamma \tilde{R}_{\alpha\beta\delta\varepsilon} + \tilde{\nabla}_\delta \tilde{R}_{\alpha\beta\varepsilon\gamma} = -T_{\gamma\delta}{}^\rho \tilde{R}_{\alpha\beta\varepsilon\rho} - T_{\delta\varepsilon}{}^\rho \tilde{R}_{\alpha\beta\gamma\rho} - T_{\varepsilon\gamma}{}^\rho \tilde{R}_{\alpha\beta\delta\rho}. \tag{2.17}$$

The tensors derived from the Riemann tensors are the Ricci tensors

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}, \qquad \tilde{R}_{\alpha\beta} = \tilde{R}^\gamma{}_{\alpha\gamma\beta}, \qquad (2.18, 2.19)$$

and the Riemann scalars

$$R = R^\alpha{}_\alpha, \qquad \tilde{R} = \tilde{R}^\alpha{}_\alpha. \qquad (2.20, 2.21)$$

Note that

$$R_{\beta\alpha} = R_{\alpha\beta} \tag{2.22}$$

but, because $\tilde{R}_{\gamma\delta\alpha\beta} \neq \tilde{R}_{\alpha\beta\gamma\delta}$,

$$\tilde{R}_{\beta\alpha} \neq \tilde{R}_{\alpha\beta}. \tag{2.23}$$

We use the Levi–Civita tensor $\varepsilon_{\alpha\beta\gamma\delta}$ which is antisymmetric on interchange of neighbouring pairs of indices. We take

$$\varepsilon_{0123} = 1, \tag{2.24}$$

and because we are in Euclidean space,

$$\varepsilon^{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta}. \tag{2.25}$$

Using $\varepsilon_{\alpha\beta\gamma\delta}$ we can define the duals of the torsion and Riemann tensors

$$^*T_{\alpha\beta}{}^\rho \equiv \tfrac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu} T_{\mu\nu}{}^\rho, \qquad ^*R_{\alpha\beta\gamma\delta} \equiv \tfrac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu} R_{\mu\nu\gamma\delta}, \qquad (2.26, 2.27)$$

$$^*\tilde{R}_{\alpha\beta\gamma\delta} \equiv \tfrac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu} \tilde{R}_{\mu\nu\gamma\delta}, \qquad \tilde{R}^*_{\alpha\beta\gamma\delta} \equiv \tfrac{1}{2}\varepsilon_{\gamma\delta}{}^{\mu\nu} \tilde{R}_{\alpha\beta\mu\nu}. \tag{2.28}$$

From these we produce the self-dual (+) or anti-self-dual (−) parts of the torsion and Riemann tensors

$$^\pm T_{\alpha\beta}{}^\rho = \tfrac{1}{2}(T_{\alpha\beta}{}^\rho \pm {}^*T_{\alpha\beta}{}^\rho), \qquad ^\pm R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(R_{\alpha\beta\gamma\delta} \pm {}^*R_{\alpha\beta\gamma\delta}), \qquad (2.29, 2.30)$$

$$\tilde{R}^\pm_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(\tilde{R}_{\alpha\beta\gamma\delta} \pm \tilde{R}^*_{\alpha\beta\gamma\delta}), \qquad ^\pm\tilde{R}_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(\tilde{R}_{\alpha\beta\gamma\delta} \pm {}^*\tilde{R}_{\alpha\beta\gamma\delta}). \tag{2.31}$$

Note that when the torsion is non-zero, the Riemann tensor has right *and* left duals and therefore right and left dual parts.

The operations of symmetrisation and antisymmetrisation on indices are respectively

$$A_{(\mu\nu)} \equiv \tfrac{1}{2}(A_{\mu\nu} + A_{\nu\mu}), \qquad A_{[\mu\nu]} \equiv \tfrac{1}{2}(A_{\mu\nu} - A_{\nu\mu}). \tag{2.32, 2.33}$$

Before moving onto two-component spinors, we give the definition of right-(+) or left-(−) handed fermions

$$\psi_\pm \equiv \tfrac{1}{2}(1 \pm \gamma_5)\psi. \tag{2.34}$$

γ_5 is the usual Dirac matrix

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.35}$$

where I is the 2×2 unit matrix. Lastly, we use the covariant d'Alembertian

$$\Box \equiv \nabla_\mu \nabla^\mu. \tag{2.36}$$

Two-component spinors provide a very nice formalism for studying the arbitrary spin problem. Tensor and spinor indices look the same and are manipulated in the same way. The rules are very simple (see Pirani (1964) and Jackiw and Rebbi (1977) for details). Each tensor index is replaced by a pair of spinor indices, one unprimed and one primed,

$$T_{\alpha\beta\ldots} \to T_{AA'BB'\ldots}, \tag{2.37}$$

(capital Latin indices take on values 1 and 2) and each spinor has either one primed or one unprimed index according to how the spinor transforms under the SO(4) group irreducible representations $(\frac{1}{2}, 0)$and $(0, \frac{1}{2})$ (Weinberg 1965). For example,

$$\psi_+ \to \psi_A, \qquad \psi_- \to \psi_{A'}. \tag{2.38}$$

A primed index can always be commuted through an unprimed index, but two primed or two unprimed indices may not be commuted unless the tensor is given some special symmetry property. In other words,

$$\phi_{AA'} = \phi_{A'A},$$

but in general it is not necessarily true that

$$\phi_{BA} = \phi_{AB} \qquad \text{or} \qquad \phi_{B'A'} = \phi_{A'B'}.$$

A general spinor $\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}$ transforms according to the (A, B) irreducible representation of SO(4) if

$$\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}} = \phi_{(A_1\ldots A_{2A})(A'_1\ldots A'_{2B})}. \tag{2.39}$$

Such a multi-spinor has

$$D(A, B) = (2A+1)(2B+1) \tag{2.40}$$

degrees of freedom. Any multi-index spinor can be transformed into a combination of spinors that transform according to different (A, B) representations. To do this we need the spinors

$$\varepsilon^{AB} = \varepsilon_{AB} = \varepsilon^{A'B'} = \varepsilon_{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.41}$$

Clearly, each of these spinors is antisymmetric. They are used to raise and lower indices according to the rules

$$\phi^{A}=\varepsilon^{AB}\phi_{B}, \qquad \phi_{A}=\phi^{B}\varepsilon_{BA},$$
$$\phi^{A'}=\varepsilon^{A'B'}\phi_{B'}, \qquad \phi_{A'}=\phi^{B'}\varepsilon_{B'A'}. \tag{2.42}$$

(These rules must be followed carefully. Sign mistakes occur if any index is out of place.) It is easy to see that

$$\delta_{A}{}^{B}=\varepsilon^{BC}\varepsilon_{AC}, \qquad \delta_{A}{}^{A}=2, \tag{2.43}$$

with the indices in precisely this order, is the Kronecker delta and that for any spinor

$$\phi_{A\ldots}{}^{A}{}_{\ldots}=-\phi^{A}{}_{\ldots A\ldots}. \tag{2.44}$$

A simple example of a spinor decomposition is that of a two-index spinor ϕ_{AB}. Write

$$\phi_{AB}=\tfrac{1}{2}(\phi_{AB}+\phi_{BA})+\tfrac{1}{2}(\phi_{AB}-\phi_{BA}).$$

Note that any antisymmetric two-index spinor must be proportional to ε_{AB}. Thus

$$\tfrac{1}{2}(\phi_{AB}-\phi_{BA})=C\varepsilon_{AB}.$$

Hitting this with ε^{AB}, we get $\phi_{A}{}^{A}=2C$ or $C=\frac{1}{2}\phi_{C}{}^{C}$.

Thus

$$\phi_{AB}=\phi_{(AB)}+\tfrac{1}{2}\varepsilon_{AB}\phi_{C}{}^{C}. \tag{2.45}$$

ϕ_{AB} contains a part that transforms via $(1,0)$, $\phi_{(AB)}$, and a part that transforms via $(0,0)$, $\phi_{C}{}^{C}$. Any multi-spinor decomposes similarly. Equation (2.45) indicates that if a spinor is symmetric in a pair of indices, it is trace-free in those indices. In the next section we decompose the torsion and Riemann tensor into irreducible spinors.

3. Spinor decompositions

The decomposition of the torsion, contorsion and Riemann tensors is a straightforward but tedious task. We will start with the simplest case, the torsion. The first step is to write

$$T_{\alpha\beta\gamma}\rightarrow T_{AA'BB'CC'}.$$

Remembering equation (2.5), we take

$$\begin{aligned}T_{AA'BB'CC'}&=\tfrac{1}{2}(T_{AA'BB'CC'}-T_{BB'AA'CC'})\\&=\tfrac{1}{2}(T_{AA'BB'CC'}-T_{BA'AB'CC'}+T_{BA'AB'CC'}-T_{BB'AA'CC'})\\&=\tfrac{1}{2}\varepsilon_{AB}T_{PA'}{}^{P}{}_{B'CC'}+\tfrac{1}{2}\varepsilon_{A'B'}T_{BP'A}{}^{P'}{}_{CC'}.\end{aligned} \tag{3.1}$$

Now consider $T_{BP'A}{}^{P'}{}_{CC'}$. Because of (2.5) and (2.44) we have

$$T_{BP'A}{}^{P'}{}_{CC'}=-T_{A}{}^{P'}{}_{BP'CC'}=T_{AP'B}{}^{P'}{}_{CC'}.$$

This says that $T_{AP'B}{}^{P'}{}_{CC'}$ and $T_{PA'}{}^{P}{}_{B'CC'}$ are symmetric in AB and $A'B'$.

We can decompose $T_{AP'B}{}^{P'}{}_{CC'}$ now. Take

$$T_{AP'B}{}^{P'}{}_{CC'} = \tfrac{1}{3}(T_{AP'B}{}^{P'}{}_{CC'} + T_{AP'C}{}^{P'}{}_{BC'} + T_{BP'C}{}^{P'}{}_{AC'}) + \tfrac{1}{3}(T_{AP'B}{}^{P'}{}_{CC'} - T_{AP'C}{}^{P'}{}_{BC'})$$
$$+ \tfrac{1}{3}(T_{AP'B}{}^{P'}{}_{CC'} - T_{BP'C}{}^{P'}{}_{AC'})$$
$$= T_{(A|P'|B|}{}^{P'}{}_{|C)C'} + \tfrac{1}{3}\varepsilon_{BC}T_{AP'Q}{}^{P'Q}{}_{C'} + \tfrac{1}{3}\varepsilon_{AC}T_{BP'Q}{}^{P'Q}{}_{C'}.$$

Define

$$T_{ABCC'} = T_{(ABC)C'} \equiv \tfrac{1}{2}T_{(A|P'|B|}{}^{P'}{}_{|C)C'}, \qquad \bar{T}_{A'B'C'C} = \bar{T}_{(A'B'C')C} \equiv \tfrac{1}{2}T_{P(A'|}{}^{P}{}_{|B'|C|C')},$$
$$U_{AC'} \equiv \tfrac{1}{6}\, T_{AP'Q}{}^{P'Q}{}_{C'}, \qquad \bar{U}_{A'C} \equiv \tfrac{1}{6}\, T_{PA'}{}^{P}{}_{Q'C}{}^{Q'}, \tag{3.2}$$

and (3.1) becomes

$$T_{AA'BB'CC'} = \varepsilon_{A'B'}(T_{ABCC'} + \varepsilon_{BC}U_{AC'} + \varepsilon_{AC}U_{BC'})$$
$$+ \varepsilon_{AB}(\bar{T}_{A'B'C'C} + \varepsilon_{B'C'}\bar{U}_{A'C} + \varepsilon_{A'C'}\bar{U}_{B'C}). \tag{3.3}$$

This is the torsion decomposition. $T_{ABCC'}, \bar{T}_{A'B'C'C}$, $U_{AC'}$ and $\bar{U}_{A'C}$ transform according to the $(\tfrac{3}{2}, \tfrac{1}{2})$, $(\tfrac{1}{2}, \tfrac{3}{2})$, $(\tfrac{1}{2}, \tfrac{1}{2})$ and $(\tfrac{1}{2}, \tfrac{1}{2})$ representations respectively. $T_{ABCC'}$ and $T_{A'B'C'C}$ each have $D = 8$ while $U_{AC'}$ and $\bar{U}_{A'C}$ have $D = 4$. So the total is $D = 24$ which is correct for $T_{\alpha\beta\gamma}$.

The decomposition of $K_{\alpha\gamma\beta}$ is similar to that for $T_{\alpha\beta\gamma}$. Using (2.8), we derive

$$K_{AA'CC'BB'} = \varepsilon_{A'B'}(K_{ABCC'} + \varepsilon_{BC}L_{AC'} + \varepsilon_{AC}L_{BC'})$$
$$+ \varepsilon_{AB}(\bar{K}_{A'B'C'C} + \varepsilon_{B'C'}\bar{L}_{A'C} + \varepsilon_{A'C'}\bar{L}_{B'C}) \tag{3.4}$$

where

$$K_{ABCC'} \equiv \tfrac{1}{2}K_{(A|P'|C|C'|B)}{}^{P'}, \qquad \bar{K}_{A'B'C'C} \equiv \tfrac{1}{2}K_{P(A'|C|C'|}{}^{P}{}_{|B')},$$
$$L_{AC'} \equiv \tfrac{1}{6}K_{AP'}{}^{Q}{}_{C'Q}{}^{P'}, \qquad \bar{L}_{A'C} \equiv \tfrac{1}{6}\bar{K}_{PA'C}{}^{Q'P}{}_{Q'}. \tag{3.5}$$

We can now translate these quantities into torsion spinors (3.2). From the definition (2.6)

$$K_{AA'CC'BB'} = \tfrac{1}{2}(T_{AA'CC'BB'} - T_{BB'AA'CC'} - T_{BB'CC'AA'}), \tag{3.6}$$

so that

$$K_{AP'CC'B}{}^{P'} = \tfrac{1}{2}(T_{AP'CC'B}{}^{P'} - T_{B}{}^{P'}{}_{AP'CC'} - T_{B}{}^{P'}{}_{CC'AP'})$$
$$= 2T_{ABCC'} + \tfrac{1}{2}\varepsilon_{BC}U_{AC}{}' + \tfrac{1}{2}\varepsilon_{AC}U_{BC'} + \tfrac{3}{2}\varepsilon_{AC}\bar{U}_{C'B} + \tfrac{3}{2}\varepsilon_{BC}\bar{U}_{C'A}$$

and

$$K_{AP'}{}^{Q}{}_{C'Q}{}^{P'} = \tfrac{3}{2}U_{AC'} + \tfrac{9}{2}\bar{U}_{C'A}.$$

Thus

$$K_{ABCC'} = T_{ABCC'}, \qquad L_{AC'} = \tfrac{1}{4}U_{AC'} + \tfrac{3}{4}\bar{U}_{C'A},$$

giving

$$K_{AA'CC'BB'} = \varepsilon_{A'B'}(T_{ABCC'} + \tfrac{1}{4}\varepsilon_{BC}U_{AC'} + \tfrac{3}{4}\varepsilon_{BC}\bar{U}_{C'A} + \tfrac{1}{4}\varepsilon_{AC}U_{BC'} + \tfrac{3}{4}\varepsilon_{AC}\bar{U}_{C'B})$$
$$+ \varepsilon_{AB}(\bar{T}_{A'B'C'C} + \tfrac{1}{4}\varepsilon_{B'C'}\bar{U}_{A'C} + \tfrac{3}{4}\varepsilon_{B'C'}U_{CA'} + \tfrac{1}{4}\varepsilon_{A'C'}\bar{U}_{B'C} + \tfrac{3}{4}\varepsilon_{A'C'}U_{CB'}). \tag{3.7}$$

Moving onto $\tilde{R}_{\alpha\beta\gamma\delta}$ we find, after a calculation similar to the two previous decompositions, that

$$\begin{aligned}\tilde{R}_{AA'BB'CC'DD'} &= \varepsilon_{A'B'}\varepsilon_{C'D'}[\tilde{\Psi}_{ABCD} + (\varepsilon_{BC}\tilde{\Theta}_{AD} + \varepsilon_{AC}\tilde{\Theta}_{BD} + \varepsilon_{BD}\tilde{\Theta}_{AC} + \varepsilon_{AD}\tilde{\Theta}_{BC}) \\ &\quad + \tilde{\Lambda}(\varepsilon_{BC}\varepsilon_{AD} + \varepsilon_{BD}\varepsilon_{AC})] + \varepsilon_{AB}\varepsilon_{CD}[\tilde{\bar{\Psi}}_{A'B'C'D'} + (\varepsilon_{B'C'}\tilde{\bar{\Theta}}_{A'D'} + \varepsilon_{A'C'}\tilde{\bar{\Theta}}_{B'D'} \\ &\quad + \varepsilon_{B'D'}\tilde{\bar{\Theta}}_{A'C'} + \varepsilon_{A'D'}\tilde{\bar{\Theta}}_{B'C'}) + \tilde{\bar{\Lambda}}(\varepsilon_{B'C'}\varepsilon_{A'D'} + \varepsilon_{B'D'}\varepsilon_{A'C'})] \\ &\quad + \varepsilon_{AB}\varepsilon_{C'D'}\tilde{\bar{\Phi}}_{A'B'CD} + \varepsilon_{A'B'}\varepsilon_{CD}\tilde{\Phi}_{ABC'D'}\end{aligned} \tag{3.8}$$

with

$$\begin{aligned}&\tilde{\Psi}_{ABCD} \equiv \tfrac{1}{4}\tilde{R}_{(A|P'|B|}{}^{P'}{}_{|C|Q'|D)}{}^{Q'}, && \tilde{\bar{\Psi}}_{A'B'C'D'} \equiv \tfrac{1}{4}\tilde{R}_{P(A'|}{}^{P}{}_{|B'|Q|C'|}{}^{Q}{}_{|D')}, \\ &\tilde{\Phi}_{ABC'D'} \equiv \tfrac{1}{4}\tilde{R}_{(A|P'|B)}{}^{P'}{}_{Q(C'|}{}^{Q}{}_{|D')}, && \tilde{\bar{\Phi}}_{A'B'CD} \equiv \tfrac{1}{4}\tilde{R}_{P(A'|}{}^{P}{}_{|B')(C|P'|D)}{}^{P'}, \\ &\tilde{\Theta}_{AB} \equiv \tfrac{1}{16}\tilde{R}_{(A|P'Q}{}^{P'Q}{}_{Q'|B)}{}^{Q'}, && \tilde{\bar{\Theta}}_{A'B'} \equiv \tfrac{1}{16}\tilde{R}_{P(A'|}{}^{P}{}_{Q'Q}{}^{Q'Q}{}_{|B')}, \\ &\tilde{\Lambda} \equiv \tfrac{1}{24}\tilde{R}_{PP'Q}{}^{P'Q}{}_{Q'}{}^{PQ'}, && \tilde{\bar{\Lambda}} \equiv \tfrac{1}{24}\tilde{R}_{PP'}{}^{P}{}_{Q'Q}{}^{Q'QP'}.\end{aligned} \tag{3.9}$$

From (3.9) we have the Ricci tensor decomposition

$$\begin{aligned}\tilde{R}_{BB'DD'} &\equiv \tilde{R}_{AA'BB'}{}^{AA'}{}_{DD'} \\ &= 4\varepsilon_{BD}\tilde{\bar{\Theta}}_{B'D'} + 4\varepsilon_{B'D'}\tilde{\Theta}_{BD} - \tilde{\Phi}_{BDB'D'} - \tilde{\bar{\Phi}}_{B'D'BD} + 3(\tilde{\Lambda} + \tilde{\bar{\Lambda}})\varepsilon_{BD}\varepsilon_{B'D'}\end{aligned} \tag{3.10}$$

and the Riemann scalar decomposition

$$\tilde{R} \equiv \tilde{R}_{AA'}{}^{AA'} = 12(\tilde{\Lambda} + \tilde{\bar{\Lambda}}). \tag{3.11}$$

Before looking at this decomposition in more detail, we write the well known $R_{\alpha\beta\gamma\delta}$ decomposition (see Pirani 1964)

$$\begin{aligned}R_{AA'BB'CC'DD'} &= \varepsilon_{A'B'}\varepsilon_{C'D'}[\Psi_{ABCD} + \Lambda(\varepsilon_{BC}\varepsilon_{AD} + \varepsilon_{BD}\varepsilon_{AC})] \\ &\quad + \varepsilon_{AB}\varepsilon_{CD}[\bar{\Psi}_{A'B'C'D'} + \Lambda(\varepsilon_{B'C'}\varepsilon_{A'D'} + \varepsilon_{B'D'}\varepsilon_{A'C'})] \\ &\quad + \varepsilon_{AB}\varepsilon_{C'D'}\Phi_{CDA'B'} + \varepsilon_{A'B'}\varepsilon_{CD}\Phi_{ABC'D'}\end{aligned} \tag{3.12}$$

with

$$\begin{aligned}&\Psi_{ABCD} \equiv \tfrac{1}{4}R_{(A|P'|B|}{}^{P'}{}_{|C|Q'|D)}{}^{Q'}, && \bar{\Psi}_{A'B'C'D'} \equiv \tfrac{1}{4}R_{P(A'|}{}^{P}{}_{|B'|Q|C'|}{}^{Q}{}_{|D')}, \\ &\Phi_{ABC'D'} \equiv \tfrac{1}{4}R_{(A|P'|B)}{}^{P'}{}_{Q(C'|}{}^{Q}{}_{|D')}, && \Lambda \equiv \tfrac{1}{24}R_{PP'Q}{}^{P'Q}{}_{Q'}{}^{PQ'},\end{aligned} \tag{3.13}$$

and then

$$R_{AA'BB'} = -2\Phi_{ABA'B'} + 6\Lambda\varepsilon_{AB}\varepsilon_{A'B'}, \qquad R = 24\Lambda. \tag{3.14, 3.15}$$

Thus

$$\Lambda = \tfrac{1}{24}R, \qquad \Phi_{ABA'B'} = -\tfrac{1}{2}(R_{AA'BB'} - \tfrac{1}{4}R\varepsilon_{AB}\varepsilon_{A'B'}), \tag{3.16}$$

$$C_{AA'BB'CC'DD'} = \varepsilon_{A'B'}\varepsilon_{C'D'}\Psi_{ABCD} + \varepsilon_{AB}\varepsilon_{CD}\bar{\Psi}_{A'B'C'D'}, \tag{3.17}$$

where $C_{AA'BB'CC'DD'}$ is the Weyl tensor. The spinor $\Phi_{ABA'B'}$ is just minus one-half the trace-free part of the Ricci tensor.

It is the loss of symmetry (2.13) that makes (3.8) more complicated than (3.12). The spinors $\tilde{\Phi}_{ABA'B'}$, $\tilde{\bar{\Phi}}_{A'B'AB}$, $\tilde{\bar{\Lambda}}$ and $\tilde{\Lambda}$ still represent the symmetric part of the Ricci tensor. The extra $\tilde{\Theta}_{AB}$ and $\tilde{\bar{\Theta}}_{A'B'}$ terms come from the antisymmetric part of $\tilde{R}_{AA'BB'}$, namely,

$$\tfrac{1}{2}(\tilde{R}_{AA'BB'} - \tilde{R}_{BB'AA'}) = 4\varepsilon_{AB}\tilde{\bar{\Theta}}_{A'B'} + 4\varepsilon_{A'B'}\tilde{\Theta}_{AB}. \tag{3.18}$$

In the next section we shall see how the quantities defined in terms of the $\tilde{\Gamma}^{\alpha}{}_{\beta\gamma}$ connection can be written in terms of objects constructed from $\Gamma^{\alpha}{}_{\beta\gamma}$ and $T_{\alpha\beta}{}^{\gamma}$.

4. Riemann tensor conversion

Ultimately we will want to be able to compare computations done with $\tilde{\Gamma}^{\alpha}{}_{\beta\gamma}$ with those using $\Gamma^{\alpha}{}_{\beta\gamma}$. In this section we will present the rather complicated relations between the $\tilde{R}_{\alpha\beta\gamma\delta}$ Riemann tensor and its spinor decomposition and $R_{\alpha\beta\gamma\delta}$ and its decomposition.

We begin with equations (2.7), (2.9) and (2.10) and find that

$$\tilde{R}^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta} + \nabla_{\gamma} K_{\beta\delta}{}^{\alpha} - \nabla_{\delta} K_{\beta\gamma}{}^{\alpha} + K_{\lambda\gamma}{}^{\alpha} K_{\beta\delta}{}^{\lambda} - K_{\lambda\delta}{}^{\alpha} K_{\beta\gamma}{}^{\lambda}, \tag{4.1}$$

$$\tilde{R}_{\beta\delta} = R_{\beta\delta} + \nabla_{\alpha} K_{\beta\delta}{}^{\alpha} - \nabla_{\delta} K_{\beta\alpha}{}^{\alpha} + K_{\lambda\alpha}{}^{\alpha} K_{\beta\delta}{}^{\lambda} - K_{\lambda\delta}{}^{\alpha} K_{\beta\alpha}{}^{\lambda}, \tag{4.2}$$

$$\tilde{R} = R + \nabla_{\alpha} K_{\beta}{}^{\beta\alpha} - \nabla^{\beta} K_{\beta\alpha}{}^{\alpha} + K_{\lambda\alpha}{}^{\alpha} K_{\beta}{}^{\beta\lambda} - K_{\lambda}{}^{\beta\alpha} K_{\beta\alpha}{}^{\lambda}. \tag{4.3}$$

These can also be written in terms of $T_{\alpha\beta}{}^{\gamma}$ rather than $K_{\alpha\gamma\beta}$ by using (2.6):

$$\begin{aligned}
\tilde{R}^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta} &+ \tfrac{1}{2}(\nabla_{\gamma} T_{\beta\delta}{}^{\alpha} - \nabla_{\delta} T_{\beta\gamma}{}^{\alpha} + \nabla_{\delta} T^{\alpha}{}_{\beta\gamma} - \nabla_{\gamma} T^{\alpha}{}_{\beta\delta} + \nabla_{\delta} T^{\alpha}{}_{\gamma\beta} - \nabla_{\gamma} T^{\alpha}{}_{\delta\beta}) \\
&+ \tfrac{1}{4}(T_{\lambda\gamma}{}^{\alpha} T_{\beta\delta}{}^{\lambda} - T_{\lambda\delta}{}^{\alpha} T_{\beta\gamma}{}^{\lambda} + T_{\lambda\delta}{}^{\alpha} T^{\lambda}{}_{\beta\gamma} - T_{\lambda\gamma}{}^{\alpha} T^{\lambda}{}_{\beta\delta} + T_{\lambda\delta}{}^{\alpha} T^{\lambda}{}_{\gamma\beta} - T_{\lambda\gamma}{}^{\alpha} T^{\lambda}{}_{\delta\beta} \\
&+ T^{\alpha}{}_{\lambda\delta} T_{\beta\gamma}{}^{\lambda} - T^{\alpha}{}_{\lambda\gamma} T_{\beta\delta}{}^{\lambda} + T^{\alpha}{}_{\lambda\gamma} T^{\lambda}{}_{\beta\delta} - T^{\alpha}{}_{\lambda\delta} T^{\lambda}{}_{\beta\gamma} + T^{\alpha}{}_{\lambda\gamma} T^{\lambda}{}_{\delta\beta} - T^{\alpha}{}_{\lambda\delta} T^{\lambda}{}_{\gamma\beta} \\
&+ T^{\alpha}{}_{\delta\lambda} T_{\beta\gamma}{}^{\lambda} - T^{\alpha}{}_{\gamma\lambda} T_{\beta\delta}{}^{\lambda} + T^{\alpha}{}_{\gamma\lambda} T^{\lambda}{}_{\beta\delta} - T^{\alpha}{}_{\delta\lambda} T^{\lambda}{}_{\beta\gamma} + T^{\alpha}{}_{\gamma\lambda} T^{\lambda}{}_{\delta\beta} - T^{\alpha}{}_{\delta\lambda} T^{\lambda}{}_{\gamma\beta}),
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\tilde{R}_{\beta\delta} = R_{\beta\delta} &+ \tfrac{1}{2}\nabla_{\alpha} T_{\beta\delta}{}^{\alpha} - \nabla_{\delta} T_{\beta\alpha}{}^{\alpha} - \tfrac{1}{2}\nabla_{\alpha} T^{\alpha}{}_{\beta\delta} - \tfrac{1}{2}\nabla_{\alpha} T^{\alpha}{}_{\delta\beta} \\
&+ \tfrac{1}{2} T_{\lambda\alpha}{}^{\alpha}(T_{\beta\delta}{}^{\lambda} - T^{\lambda}{}_{\beta\delta} - T^{\lambda}{}_{\delta\beta}) + \tfrac{1}{4} T^{\alpha\lambda}{}_{\beta} T_{\alpha\lambda\delta} + \tfrac{1}{2} T^{\alpha\lambda}{}_{\beta} T_{\alpha\delta\lambda},
\end{aligned} \tag{4.5}$$

$$\tilde{R} = R - 2\nabla^{\alpha} T_{\alpha\beta}{}^{\beta} - T_{\lambda\alpha}{}^{\alpha} T^{\lambda}{}_{\beta}{}^{\beta} + \tfrac{1}{4} T^{\alpha\beta\lambda} T_{\alpha\beta\lambda} + \tfrac{1}{2} T^{\alpha\beta\lambda} T_{\alpha\lambda\beta}. \tag{4.6}$$

Either (4.1) or (4.4) can be used to find the spinor quantities with a tilde on them in terms of those with no tilde and the torsion tensor decomposition. It turns out that using (4.1) shortens the work. It is easy to derive the relations

$$\tilde{\Psi}_{ABCD} = \Psi_{ABCD} + \nabla^{C'}{}_{(A} T_{BCD)C'} - T_{PP'(AB} T_{CD)}{}^{PP'} + \tfrac{1}{2} T_{C'(ABC} U_{D)}{}^{C'} + \tfrac{3}{2} T_{C'(ABC} \bar{U}_{D)}{}^{C'}, \tag{4.7}$$

$$\tilde{\Theta}_{AB} = \tfrac{1}{8}\nabla^{CC'} T_{ABCC'} + \tfrac{1}{8}\nabla^{C'}{}_{(A} U_{B)C'} + \tfrac{3}{8}\nabla^{C'}{}_{(A} \bar{U}_{B)C'} - \tfrac{3}{16} T_{ABPC'} U^{PC'} - \tfrac{9}{16} T_{ABPC'} \bar{U}^{PC'}, \tag{4.8}$$

$$\begin{aligned}
\tilde{\Lambda} = \Lambda &+ \tfrac{1}{8}\nabla_{CC'} U^{CC'} + \tfrac{3}{8}\nabla_{CC'} \bar{U}^{CC'} - \tfrac{1}{16} U_{CC'} U^{CC'} - \tfrac{9}{16} \bar{U}_{CC'} \bar{U}^{CC'} \\
&- \tfrac{3}{8} U_{CC'} \bar{U}^{CC'} + \tfrac{1}{12} T_{ABCC'} T^{ABCC'},
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\tilde{\Phi}^{AB}{}_{C'D'} = \Phi^{AB}{}_{C'D'} &+ \nabla^{C}{}_{(C'} T_{D')C}{}^{AB} - T_{PQ}{}^{(A}{}_{(C'} T_{D')}{}^{B)PQ} - \tfrac{1}{2} T^{AB}{}_{P(C'} U_{D')}{}^{P} - \tfrac{3}{2} T^{AB}{}_{P(C'} \bar{U}_{D')}{}^{P} \\
&- \tfrac{1}{2}\nabla^{(A}{}_{(C'} U_{D')}{}^{B)} - \tfrac{3}{2}\nabla^{(A}{}_{(C'} \bar{U}_{D')}{}^{B)} - \tfrac{1}{4} U^{(A}{}_{(C'} U_{D')}{}^{B)} \\
&- \tfrac{9}{4} \bar{U}^{(A}{}_{(C'} \bar{U}_{D')}{}^{B)} - \tfrac{3}{2} U^{(A}{}_{(C'} \bar{U}_{D')}{}^{B)}.
\end{aligned} \tag{4.10}$$

The other four quantities $\tilde{\bar{\Psi}}_{A'B'C'D'}$, $\tilde{\bar{\Theta}}_{A'B'}$, $\tilde{\bar{\Lambda}}$ and $\tilde{\bar{\Phi}}^{A'B'}{}_{CD}$ can be obtained from (4.7)–(4.10) by making every primed index unprimed, every unprimed index primed, putting bars on all the T's and on Ψ_{ABCD}, taking off the bars of the $\bar{U}$'s and putting them on the U's. For example

$$\tilde{\tilde{\Lambda}} = \Lambda + \tfrac{1}{8}\nabla_{CC'}\bar{U}^{CC'} + \tfrac{3}{8}\nabla_{CC'}U^{CC'} - \tfrac{1}{16}\bar{U}_{CC'}\bar{U}^{CC'} - \tfrac{9}{16}U_{CC'}U^{CC'} - \tfrac{3}{8}U_{CC'}\bar{U}^{CC'} + \bar{T}_{A'B'C'C}\bar{T}^{A'B'C'C}.$$

(In deriving the above we use the property of two-component spinors that says $L_{C'(A\ldots B}L_{C\ldots D)}{}^{C'} = 0$ for any spinor $L_{C'A\ldots B}$.)

5. Spinor Ricci identity with torsion

We now generalise the spinorial version of the Ricci identity given by Pirani (1964) to the non-zero torsion case (remembering that we use Misner *et al* (1973) conventions which are slightly different from Pirani's). In tensor form, the Ricci identity for the covariant derivative $\tilde{\nabla}_\mu$ acting on the tensor $B_{\alpha\beta}$ is

$$(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \tilde{\nabla}_\nu\tilde{\nabla}_\mu)B_{\alpha\beta} = T_{\mu\nu}{}^{\rho}\tilde{\nabla}_\rho B_{\alpha\beta} + \tilde{R}_\alpha{}^{\rho}{}_{\mu\nu}B_{\rho\beta} + \tilde{R}_\beta{}^{\rho}{}_{\mu\nu}B_{\alpha\rho}. \tag{5.1}$$

This is easily obtained from the definition of $\tilde{\nabla}_\mu$ acting on $B_{\alpha\beta}$

$$\tilde{\nabla}_\mu B_{\alpha\beta} \equiv \partial_\mu B_{\alpha\beta} - \tilde{\Gamma}^{\rho}{}_{\alpha\mu}B_{\rho\beta} - \tilde{\Gamma}^{\rho}{}_{\beta\mu}B_{\alpha\rho}. \tag{5.2}$$

In spinor terms equation (5.1) becomes

$$(\tilde{\nabla}_{AA'}\tilde{\nabla}_{BB'} - \tilde{\nabla}_{BB'}\tilde{\nabla}_{AA'})B_{CC'DD'} = T_{AA'BB'PP'}\tilde{\nabla}^{PP'}B_{CC'DD'} + \tilde{R}_{CC'}{}^{PP'}{}_{AA'BB'}B_{PP'DD'} + \tilde{R}_{DD'}{}^{PP'}{}_{AA'BB'}B_{CC'PP'}. \tag{5.3}$$

Following Pirani, we can write

$$\tilde{\nabla}_{AA'}\tilde{\nabla}_{BB'} - \tilde{\nabla}_{BB'}\tilde{\nabla}_{AA'} = \varepsilon_{AB}\tilde{\nabla}_{P(A'}\tilde{\nabla}^{P}{}_{B')} + \varepsilon_{A'B'}\tilde{\nabla}_{P'(A}\tilde{\nabla}^{P'}{}_{B)} \tag{5.4}$$

and choose

$$B_{CC'DD'} = \xi_C\xi_D\varepsilon_{C'D'}. \tag{5.5}$$

Equation (5.3) becomes

$$\varepsilon_{AB}\tilde{\nabla}_{P(A'}\tilde{\nabla}^{P}{}_{B')}\xi_C\xi_D\varepsilon_{C'D'} + \varepsilon_{A'B'}\tilde{\nabla}_{P'(A}\tilde{\nabla}^{P'}{}_{B)}\xi_C\xi_D\varepsilon_{C'D'} = T_{AA'BB'PP'}\tilde{\nabla}^{PP'}\xi_C\xi_D\varepsilon_{C'D'} - \tilde{R}_{CC'PD'AA'BB'}\xi^{P}\xi_D - \tilde{R}_{DD'PC'AA'BB'}\xi^{P}\xi_C$$

and, if we contract this with $\varepsilon^{AB}\varepsilon^{C'D'}$, we get

$$4\tilde{\nabla}_{P(A'}\tilde{\nabla}^{P}{}_{B')}\xi_C\xi_D = 2T_{QA'}{}^{Q}{}_{B'PP'}\tilde{\nabla}^{PP'}\xi_C\xi_D - \tilde{R}_{CQ'P}{}^{Q'}{}_{QA'}{}^{Q}{}_{B'}\xi^{P}\xi_D - \tilde{R}_{DQ'P}{}^{Q'}{}_{QA'}{}^{Q}{}_{B'}\xi^{P}\xi_C.$$

Suppose that we then contract with $\eta^C\eta^D$, where η^C is an arbitrary covariantly constant spinor, i.e.

$$\tilde{\nabla}_{AA'}\eta^C = 0.$$

It is then a simple matter to obtain

$$\tilde{\nabla}_{P(A'}\tilde{\nabla}^{P}{}_{B')}\xi_C = \tfrac{1}{2}T_{QA'}{}^{Q}{}_{B'PP'}\tilde{\nabla}^{PP'}\xi_C - \tfrac{1}{4}\tilde{R}_{CQ'P}{}^{Q'}{}_{QA'}{}^{Q}{}_{B'}\xi^{P}. \tag{5.6}$$

In a similar fashion we can also obtain

$$\tilde{\nabla}_{P'(A}\tilde{\nabla}^{P'}{}_{B)}\xi_C = \tfrac{1}{2}T_{AQ'B}{}^{Q'}{}_{PP'}\tilde{\nabla}^{PP'}\xi_C - \tfrac{1}{4}\tilde{R}_{CQ'P}{}^{Q'}{}_{AP'B}{}^{P'}\xi^{P}. \tag{5.7}$$

These are the spinor Ricci identities for a spinor with one unprimed index. For a general spinor we have

$$\begin{aligned}
&\tilde\nabla_{P'(A}\tilde\nabla_{B)}{}^{P'}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}\\
&\quad=\tfrac12 T_{AP'B}{}^{P'}{}_{CC'}\tilde\nabla^{CC'}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}-\tfrac14\tilde R_{A_1P'Q}{}^{P'}{}_{AQ'B}{}^{Q'}\phi^{Q}{}_{A_2\ldots A_{2A}A'_1\ldots A'_{2B}}\\
&\quad\quad-\ldots-\tfrac14\tilde R_{A_{2A}P'Q}{}^{P'}{}_{AQ'B}{}^{Q'}\phi_{A_1\ldots A_{2A-1}}{}^{Q}{}_{A'_1\ldots A'_{2B}}\\
&\quad\quad-\tfrac14\tilde R_{PA'_1}{}^{P}{}_{Q'AP'B}{}^{P'}\phi_{A_1\ldots A_{2A}}{}^{Q'}{}_{A'_2\ldots A'_{2B}}\\
&\quad\quad-\ldots-\tfrac14\tilde R_{PA'_{2B}}{}^{P}{}_{Q'AP'B}{}^{P'}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B-1}}{}^{Q'},
\end{aligned}\tag{5.8}$$

$$\begin{aligned}
&\tilde\nabla_{P(A'}\tilde\nabla_{B')}{}^{P}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}\\
&\quad=\tfrac12 T_{PA'}{}^{P}{}_{B'CC'}\tilde\nabla^{CC'}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}-\tfrac14\tilde R_{A_1P'Q}{}^{P'}{}_{PA'}{}^{P}{}_{B'}\phi^{Q}{}_{A_2\ldots A_{2A}A'_1\ldots A'_{2B}}\\
&\quad\quad-\ldots-\tfrac14\tilde R_{A_{2A}P'Q}{}^{P'}{}_{PA'}{}^{P}{}_{B'}\phi_{A_1\ldots A_{2A-1}}{}^{Q}{}_{A'_1\ldots A'_{2B}}\\
&\quad\quad-\tfrac14\tilde R_{PA'_1}{}^{P}{}_{Q'QA'}{}^{Q}{}_{B'}\phi_{A_1\ldots A_{2A}}{}^{Q'}{}_{A'_2\ldots A'_{2B}}\\
&\quad\quad-\ldots-\tfrac14\tilde R_{PA'_{2B}}{}^{P}{}_{Q'QA'}{}^{Q}{}_{B'}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B-1}}{}^{Q'}.
\end{aligned}\tag{5.9}$$

These identities reduce to the usual ones in the $T_{\alpha\beta}{}^{\gamma}=0$ case. Equations (5.8) and (5.9) along with the spinor decompositions of § 4 will be crucial to the analysis of the consistency conditions in § 7.

6. Arbitrary spin field equations

As we mentioned in the Introduction, there may be many different ways to write arbitrary spin field equations. Clearly we must have some criteria that permit us to choose a particular set. What restrictions should we pick? In Christensen and Duff (1979) a very natural set was presented. The primary restrictions require that the well known field equations for low spins come out of the arbitrary spin equations. Beyond that, we ask that the equations have a 'simple' form. Further, we want bosons and fermions to obey similar equations. This last restriction seems natural when investigating supersymmetric or 'hypersymmetric' theories where bosons and fermions are treated as parts of a multiplet. In this work we will choose the non-zero torsion generalisation of the restrictions and field equations presented in the paper above.

We require that the field operator $\tilde\Delta(A,B)$ acting on the field $\phi(A,B)$ satisfy the following criteria.

(1) $\tilde\Delta$ is self-adjoint, i.e.

$$(\phi,\tilde\Delta\phi)=(\tilde\Delta\phi,\phi),$$

where

$$(\phi,\psi)\equiv\int d^4x\,g^{1/2}\phi^{\dagger}\psi.$$

(2) For a scalar field ϕ, $\tilde\Delta$ is the Klein–Gordon operator

$$\tilde\Delta\phi=-\tilde\Box\phi.\tag{6.1}$$

(At this point we want to point out a potential bit of confusion in some of the literature. Kimura (1981) makes the statement that the scalar field does not couple to the torsion. It is true that (6.1) contains no explicit torsion terms, but there is torsion in the covariant derivatives. This is seen if we rewrite (6.1) as

$$-\tilde{\Box}\phi = -\Box\phi - T_{\nu\mu}{}^{\mu}\nabla^{\nu}\phi.$$

If we restrict the torsion to be trace-free then indeed there is no coupling, but in general the coupling does exist and will contribute to the one-loop counterterms, for example.)

(3) Acting on ϕ_A and $\phi_{A'}$ (from equation (2.34)), $\tilde{\Delta}$ coincides with the square of the Dirac operator $\gamma^{\mu}\tilde{\nabla}_{\mu}$. In two-component spinor language we have

$$\gamma^{\mu}\tilde{\nabla}_{\mu} \to \tilde{\nabla}_{AA'}. \tag{6.2}$$

The Dirac equation for ϕ_A is then

$$\tilde{\nabla}_B{}^{A'}\phi^B = 0. \tag{6.3}$$

We 'square' the equation by hitting (6.3) with $\tilde{\nabla}_{AA'}$

$$\tilde{\nabla}_{AA'}\tilde{\nabla}^{A'}{}_B\phi^B = 0. \tag{6.4}$$

We can rewrite this as

$$\tilde{\nabla}_{H'(A}\tilde{\nabla}_{B)}{}^{H'}\phi^B + \tilde{\nabla}_{H'[A}\tilde{\nabla}_{B]}{}^{H'}\phi^B = 0$$

or

$$-\tilde{\Box}\phi_A + 2\tilde{\nabla}_{H'(A}\tilde{\nabla}_{B)}{}^{H'}\phi^B = 0. \tag{6.5}$$

Writing out the Ricci identity part of (6.5), we have

$$-\tilde{\Box}\phi_A + T_{AQ'B}{}^{Q'}{}_{PP'}\tilde{\nabla}^{PP'}\phi^B + \tfrac{1}{2}\tilde{R}_{BP'P}{}^{P'}{}_{AQ'}{}^{BQ'}\phi^P = 0. \tag{6.6}$$

This reduces to the usual field equation when $T_{\alpha\beta\gamma} = 0$, namely,

$$-\Box\phi_A + \tfrac{1}{4}R\phi_A = 0. \tag{6.7}$$

(4) Acting on vector fields that satisfy $\tilde{\nabla}^{\mu}A_{\mu} = 0$, that is $\phi(\frac{1}{2}, \frac{1}{2})$ fields, $\tilde{\Delta}$ is

$$-\tilde{\Box}A_{\mu} - T_{\mu\nu}{}^{\rho}\tilde{\nabla}_{\rho}A^{\nu} + \tilde{R}_{\nu\mu}A^{\nu} = 0, \tag{6.8}$$

the Maxwell operator. (Remember that the Ricci tensor is no longer symmetric.)

(5) For spin-$\frac{3}{2}$ fermions that satisfy $\gamma^{\mu}\psi_{\mu} = 0$ (i.e. $\phi(1, \frac{1}{2})$ and $\phi(\frac{1}{2}, 1)$), we require that $\tilde{\Delta}$ coincides with the right- or left-handed parts of the square of the Rarita–Schwinger operator. In fact, when $\gamma^{\mu}\psi_{\mu} = 0$, we will see that $\tilde{\Delta}$ reduces to the square of the Dirac operator. The first-order spin-$\frac{3}{2}$ field equation is

$$\varepsilon^{\mu\nu\lambda\tau}\gamma_5\gamma_{\nu}\tilde{\nabla}_{\lambda}\psi_{\tau} = 0. \tag{6.9}$$

Using the identity

$$\varepsilon^{\mu\nu\lambda\tau}\gamma_5\gamma_{\nu} = \tfrac{1}{2}(\gamma^{\mu}\gamma^{\lambda}\gamma^{\tau} - \gamma^{\tau}\gamma^{\lambda}\gamma^{\mu}) = (\gamma^{\mu}\gamma^{\lambda}\gamma^{\tau} - g^{\mu\lambda}\gamma^{\tau} + g^{\mu\tau}\gamma^{\lambda} - g^{\lambda\tau}\gamma^{\mu}), \tag{6.10}$$

(6.9) becomes

$$(\gamma^{\mu}\gamma^{\lambda}\gamma^{\tau} - g^{\mu\lambda}\gamma^{\tau} + g^{\mu\tau}\gamma^{\lambda} - g^{\lambda\tau}\gamma^{\mu})\tilde{\nabla}_{\lambda}\psi_{\tau} = 0. \tag{6.11}$$

Before proceeding any further, it is important that we consider the quantity $\tilde{\nabla}_{\mu}\gamma_{\nu}$, that is, we want to know if we can commute the γ_{μ}'s through covariant derivatives. The answer is yes, *if* we define the spin connection $\tilde{\omega}^a{}_{b\mu}$ in the right way. First note that the spin connection in the torsion-free case is usually defined by requiring (see for example DeWitt 1965)

$$\nabla_\mu e^{a\nu} = \partial_\mu e^{a\nu} + \Gamma^{\nu}{}_{\rho\mu} e^{a\rho} + \omega^{a}{}_{b\mu} e^{b\nu} = 0. \tag{6.12}$$

In (6.12), $e^{a\nu}$ is the vierbein with letters from the first part of the lower case Latin alphabet indicating bein indices and letters from the later part of the Greek alphabet representing tensor indices. Clearly for (6.12) to be true we must have

$$\omega^{a}{}_{\beta\mu} = -e_{b\nu}\partial_\mu e^{a\nu} - \Gamma^{\sigma}{}_{\nu\mu} e^{a\nu} e_{b\sigma}. \tag{6.13}$$

We use this to define the covariant derivative on spinors via

$$\nabla_\mu \psi^\alpha = \partial_\mu \psi^\alpha + [\Sigma_{ab}]^\alpha{}_\beta \omega^{ab}{}_\mu \psi^\beta, \tag{6.14}$$

where we put in the spinor indices $\alpha, \beta, \ldots$ when needed. In (6.14) the SO(4) generators $[\Sigma_{ab}]^\alpha{}_\beta$ are given by

$$[\Sigma_{ab}]^\alpha{}_\beta \equiv \tfrac{1}{8}([\gamma_a]^\alpha{}_\gamma [\gamma_b]^\gamma{}_\beta - [\gamma_b]^\alpha{}_\gamma [\gamma_a]^\gamma{}_\beta). \tag{6.15}$$

Now consider $\nabla_\mu \gamma_a$. This is

$$\nabla_\mu [\gamma_a]^\alpha{}_\beta = \partial_\mu [\gamma_a]^\alpha{}_\beta - \omega_a{}^b{}_\mu [\gamma_b]^\alpha{}_\beta + [\Sigma_{cd}]^\alpha{}_\gamma \omega^{cd}{}_\mu [\gamma_a]^\gamma{}_\beta - \omega^{cd}{}_\mu [\gamma_a]^\alpha{}_\gamma [\Sigma_{cd}]^\gamma{}_\beta,$$

or since the γ_a matrices are constants, i.e. $\partial_\mu \gamma_a = 0$,

$$\nabla_\mu \gamma_a = -\omega_a{}^b{}_\mu \gamma_b + \omega^{cd}{}_\mu (\Sigma_{cd}\gamma_a - \gamma_a \Sigma_{cd}) = \omega^{cd}{}_\mu [-\delta_{ac}\gamma_d + \Sigma_{cd}\gamma_a - \gamma_a \Sigma_{cd}]. \tag{6.16}$$

Using

$$\Sigma_{cd}\gamma_a - \gamma_a \Sigma_{cd} = \tfrac{1}{2}(\delta_{ac}\gamma_d - \delta_{ad}\gamma_c), \tag{6.17}$$

(6.16) becomes

$$\nabla_\mu \gamma_a = \omega^{cd}{}_\mu [-\tfrac{1}{2}\delta_{ac}\gamma_d - \tfrac{1}{2}\delta_{ad}\gamma_c] = 0. \tag{6.18}$$

Finally, since $\gamma_\mu \equiv \gamma_a e^a{}_\mu$, and due to properties (6.12) and (6.18), we have

$$\nabla_\mu \gamma_\nu = 0. \tag{6.19}$$

Generalising (6.19) to the torsion case is easy. If we demand $\tilde{\nabla}_\mu e^{a\nu} \equiv 0$, then

$$\tilde{\omega}^{ab}{}_\mu \equiv -e^b{}_\nu \partial_\mu e^{a\nu} - \tilde{\Gamma}^\sigma{}_{\nu\mu} e^{a\nu} e^b{}_\sigma, \tag{6.20}$$

so that

$$\tilde{\omega}^{ab}{}_\mu = \omega^{ab}{}_\mu - K_{\nu\mu\sigma} e^{a\nu} e^{b\sigma}. \tag{6.21}$$

Clearly,

$$\tilde{\omega}^{ab}{}_\mu = -\tilde{\omega}^{ba}{}_\mu, \tag{6.22}$$

and so the same argument as given above results in

$$\tilde{\nabla}_\mu \gamma_\nu = 0. \tag{6.23}$$

(Note that in the spin-$\frac{1}{2}$ case, we also need (6.23) to justify squaring the Dirac operator as we did in (6.4).)

Now we go back to the analysis of (6.11). The first and second terms vanish due to (6.23) and the $\gamma^\mu \psi_\mu = 0$ condition, so that

$$\gamma^\lambda \tilde{\nabla}_\lambda \psi^\mu - \gamma^\mu \tilde{\nabla}_\lambda \psi^\lambda = 0.$$

Contract this with γ_μ and use

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\, g_{\mu\nu}\mathbb{1} \tag{6.24}$$

to obtain

$$\tilde{\nabla}_\lambda \psi^\lambda = 0. \tag{6.25}$$

Thus, in the end, (6.11) becomes

$$\gamma^\lambda \tilde{\nabla}_\lambda \psi^\mu = 0. \tag{6.26}$$

In two-component spinor language, the right-handed part of (6.26) gives

$$\tilde{\nabla}_{B'}{}^{H}\phi_{HBA'} = 0 \tag{6.27}$$

and the left-handed part gives

$$\tilde{\nabla}_{B}{}^{H'}\phi_{AH'B'} = 0. \tag{6.28}$$

We can 'square' (6.27) as we did for (6.3) to get

$$\tilde{\nabla}_A{}^{B'}\tilde{\nabla}_{B'H}\phi^H{}_{BA'} = 0$$

or

$$(\tilde{\nabla}_{(A}{}^{B'}\tilde{\nabla}_{H)B'} + \tilde{\nabla}_{[A}{}^{B'}\tilde{\nabla}_{H]B'})\phi^H{}_{BA'} = 0.$$

This gives

$$-\tilde{\Box}\phi_{ABA'} + 2\tilde{\nabla}_{H'(A}\tilde{\nabla}_{H)}{}^{H'}\phi^H{}_{BA'} = 0, \tag{6.29}$$

which after using the Ricci identities (5.8) and (5.9) is

$$-\tilde{\Box}\phi_{ABA'} + T_{AP'H}{}^{P'}{}_{CC'}\tilde{\nabla}^{CC'}\phi^H{}_{BA'} + \tfrac{1}{2}\tilde{R}_{HP'Q}{}^{P'}{}_{AQ'}{}^{HQ'}\phi^Q{}_{BA'} - \tfrac{1}{2}\tilde{R}_{BP'Q}{}^{P'}{}_{AQ'H}{}^{Q'}\phi^{HQ}{}_{A'}$$
$$-\tfrac{1}{2}\tilde{R}_{PA'}{}^{P}{}_{Q'AP'H}{}^{P'}\phi^H{}_B{}^{Q'} = 0. \tag{6.30}$$

Equation (6.30) may be written in even more detail if we substitute in the decompositions (3.2) and (3.9).

Summarising, we demand that fields $\phi(1, \frac{1}{2})$ and $\phi(\frac{1}{2}, 1)$ satisfy

$$-\tilde{\Box}\phi_{ABA'} + 2\tilde{\nabla}_{H'(A}\tilde{\nabla}_{H)}{}^{H'}\phi^H{}_{BA'} = 0, \tag{6.29}$$

$$-\tilde{\Box}\phi_{AA'B'} + 2\tilde{\nabla}_{H(A'}\tilde{\nabla}_{H')}{}^{H}\phi_A{}^{H'}{}_{B'} = 0, \tag{6.31}$$

respectively.

(6) The symmetric spin-2 field $h_{\mu\nu}$, obeying the condition $\tilde{\nabla}^\mu(h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h^\sigma{}_\sigma) = 0$, transforms according to the direct sum $(1,1)\oplus(0,0)$ representation, is governed by

$$-\tilde{\Box}h_{\mu\nu} - T_{\mu\rho}{}^{\tau}\tilde{\nabla}_\tau h^\rho{}_\nu - T_{\nu\rho}{}^{\tau}\tilde{\nabla}_\tau h_\mu{}^\rho + \tilde{R}_{\rho\mu}h^\rho{}_\nu + \tilde{R}_{\rho\nu}h_\mu{}^\rho - \tilde{R}_{\mu\rho\nu\tau}h^{\rho\tau} - \tilde{R}_{\nu\rho\mu\tau}h^{\rho\tau} = 0, \tag{6.32}$$

which is the simplest generalisation of the usual spin-2 equation

$$-\Box h_{\mu\nu} + R_{\rho\mu}h^\rho{}_\nu + R_{\rho\nu}h_\mu{}^\rho - 2R_{\mu\rho\nu\tau}h^{\rho\tau} = 0, \tag{6.33}$$

with the first derivative terms added in just the way they were in the spin-1 case. (It is interesting to ask how we might derive the spin-2 equations from an action principle. Equation (6.33) comes easily from the second functional derivative of Einstein's action. However, (6.32) is not so easily derived. We shall return to this question in a future publication.)

(7) Consider differential p-forms

$$\omega = (1/p!)\omega_{\mu_1\ldots\mu_p}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \tag{6.34}$$

(i.e. $\phi(0,0), \phi(\frac{1}{2},\frac{1}{2}), \phi(1,0)$ and $\phi(0,1)$); $\tilde{\Delta}$ for forms coincides with the non-zero torsion version of the Hodge–de Rham operator

$$\tilde{\Delta} \equiv \tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d}, \tag{6.35}$$

where

$$\tilde{d}\omega \equiv (1/p!)\tilde{\nabla}_\mu \omega_{\mu_1\ldots\mu_p}\, dx^\mu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \tag{6.36}$$

$$\tilde{\delta}\omega \equiv [1/(p-1)!]\tilde{\nabla}^\mu \omega_{\mu\mu_2\ldots\mu_p} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}. \tag{6.37}$$

This is consistent with requirements (2) and (4).

(8) We define a 'fermionic' differential p-form by putting a spinor index on the regular 'bosonic' p-forms

$$\omega^\alpha \equiv (1/p!)\omega^\alpha{}_{\mu_1\ldots\mu_p}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{6.38}$$

We also devise a new operation

$$(\gamma\omega)^\alpha = \mathrm{i}[1/(p-1)!][\gamma^\mu]^\alpha{}_\beta \omega^\beta{}_{\mu\mu_2\ldots\mu_p}\, dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}. \tag{6.39}$$

The fields $\phi(\frac{1}{2},0)$, $\phi(0,\frac{1}{2})$, $\phi(1,\frac{1}{2})$, $\phi(\frac{1}{2},1)$, $\phi(\frac{3}{2},0)$ and $\phi(0,\frac{3}{2})$ can all be represented as fermionic forms. Introducing the operator

$$\tilde{\nabla} \equiv \tilde{d}\gamma + \gamma\tilde{d}, \tag{6.40}$$

we see that for a 'fermionic 0-form'

$$(\tilde{\nabla}\omega)^\alpha = (\gamma\, \tilde{d}\omega)^\alpha = \mathrm{i}[\gamma^\mu]^\alpha{}_\beta \tilde{\nabla}_\mu \omega^\beta. \tag{6.41}$$

$\tilde{\nabla}$ is the Dirac operator for spin-$\frac{1}{2}$ fields. For spin-$\frac{3}{2}$, $\tilde{\nabla}$ is still the Dirac operator when $(\gamma\omega)^\alpha = 0$.

We require that on fermionic forms $\tilde{\Delta}$ be $\tilde{\nabla}^2$. This is consistent with requirements (3) and (5).

If we now study these eight criteria carefully we notice some patterns. Both fermions and bosons have 'form' structures. The boson operators all have a similar structure when they are written in tensor notation, as do the fermions in both tensor and two-component spinor notation. Can we make a choice of arbitrary spin operator that is simple, but also incorporates each field equation?

There is an obvious choice of $\tilde{\Delta}$ for fermions, namely

$$-\tilde{\Box}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}} + 2\tilde{\nabla}_{H'(A_1}\tilde{\nabla}_{H)}{}^{H'}\phi^H{}_{A_2\ldots A_{2A}A'_1\ldots A'_{2B}}, \qquad A > B, \tag{6.42}$$

$$-\tilde{\Box}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}} + 2\tilde{\nabla}_{H(A'_1}\tilde{\nabla}_{H')}{}^{H}\phi_{A_1\ldots A_{2A}}{}^{H'}{}_{A'_2\ldots A'_{2B}}, \qquad A < B. \tag{6.43}$$

But what about bosons? To get a two-component spinor version of the boson equations, we first look at the so-called Lichnerowicz operator acting on tensors of arbitrary rank and generalised to the non-zero torsion case

$$\tilde{\Delta}T^{\mu_1\cdots\mu_n} = -\tilde{\Box}T^{\mu_1\cdots\mu_n} - \sum_{i=1}^{n} [\tilde{\nabla}^{\mu_i}, \tilde{\nabla}_\nu] T^{\mu_1\cdots\nu\cdots\mu_n}. \tag{6.44}$$

When there are no indices,

$$\tilde{\Delta}T = -\tilde{\Box}T,$$

the scalar field operator of requirement (2). With one index,

$$\tilde{\Delta}T^{\mu_1} = -\tilde{\Box}T^{\mu_1} - [\tilde{\nabla}^{\mu_1}, \tilde{\nabla}_\nu]T^\nu = -\tilde{\Box}T^{\mu_1} - T^{\mu_1}{}_\nu{}^\rho\tilde{\nabla}_\rho T^\nu + \tilde{R}_\nu{}^{\mu_1}T^\nu, \qquad (6.45)$$

the spin-1 operator of (3). With two symmetric indices,

$$\begin{aligned}\tilde{\Delta}T^{\mu_1\mu_2} &= -\tilde{\Box}T^{\mu_1\mu_2} - [\tilde{\nabla}^{\mu_1}, \tilde{\nabla}_\nu]T^{\nu\mu_2} - [\tilde{\nabla}^{\mu_2}, \tilde{\nabla}_\nu]T^{\mu_1\nu} \\ &= -\tilde{\Box}T^{\mu_1\mu_2} - T^{\mu_1}{}_\nu{}^\rho\tilde{\nabla}_\rho T^{\nu\mu_2} - T^{\mu_2}{}_\nu{}^\rho\tilde{\nabla}_\rho T^{\mu_1\nu} \\ &\quad + \tilde{R}_\nu{}^{\mu_1}T^{\nu\mu_2} + \tilde{R}_\nu{}^{\mu_2}T^{\mu_1\nu} - T^{\nu\rho}\tilde{R}^{\mu_2}{}_\nu{}^{\mu_1}{}_\rho - T^{\nu\rho}\tilde{R}^{\mu_1}{}_\nu{}^{\mu_2}{}_\rho, \qquad (6.46)\end{aligned}$$

the spin-2 operator we want in requirement (5).

We can rewrite (6.46) in two-component spinor form using the Ricci identities (5.8) and (5.9). The result is

$$\begin{aligned}\tilde{\Delta}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}} &= -\tilde{\Box}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}} + \sum_{i=1}^{2A}\tilde{\nabla}_{H'(A_i}\tilde{\nabla}_{H)}{}^{H'}\phi_{A_1\ldots}{}^{H}{}_{\ldots A_{2A}A'_1\ldots A'_{2B}} \\ &\quad + \sum_{i=1}^{2B}\tilde{\nabla}_{H(A'_i}\tilde{\nabla}_{H')}{}^{H}\phi_{A_1\ldots A_{2A}A'_1\ldots}{}^{H'}{}_{\ldots A'_{2B}}, \qquad A+B = \text{integer}. \qquad (6.47)\end{aligned}$$

For reasons we shall give in the next section, we choose a set of boson equations which coincide with (6.47) for low spins, but differ at high spins. We pick instead

$$\begin{aligned}\tilde{\Delta}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}} &= -\tilde{\Box}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}} + 2A\tilde{\nabla}_{H'(A_1}\tilde{\nabla}_{H)}{}^{H'}\phi^{H}{}_{A_2\ldots A_{2A}A'_1\ldots A'_{2B}} \\ &\quad + 2B\tilde{\nabla}_{H(A'_1}\tilde{\nabla}_{H')}{}^{H}\phi_{A_1\ldots A_{2A}}{}^{H'}{}_{A'_2\ldots A'_{2B}}, \qquad A+B = \text{integer}. \qquad (6.48)\end{aligned}$$

This boson equation is as similar to (6.42) and (6.43) as we can make it and still have it satisfy our criteria.

In conclusion, we will use (6.42), (6.43) and (6.48) as our arbitrary spin field equations.

7. Consistency conditions

Consider the fermion field equations obtained by setting (6.42) equal to zero. Remembering that $\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}$ is completely symmetric, it is easy to see that if we hit (6.42) with $\varepsilon^{A_1A_2}$ we get

$$\tilde{\nabla}_{H'(A_1}\tilde{\nabla}_{A_2)}{}^{H'}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}A'_1\ldots A'_{2B}} = 0, \qquad A > B. \qquad (7.1)$$

This is the consistency condition for fermion fields $\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}$ with $A > B$. We notice two things immediately. First, if we have $A < 1$ there are no consistency conditions. Clearly, in order to form (7.1) in the first place, we need at least two unprimed indices on the spinor field. Secondly, we see that (7.1) is the double contraction of the Ricci identity (5.8).

Let us look at the simplest case, $(\frac{1}{2}, 0)$. Since $A = \frac{1}{2} < 1$, there is no consistency condition. Thus, so long as the manifold admits a spin structure, there will be no restriction on the torsion or curvature. However, moving on to the $(\frac{3}{2}, 0)$ case, we have

$$\tilde{\nabla}_{H'(A_1}\tilde{\nabla}_{A_2)}{}^{H'}\phi^{A_1A_2}{}_{A_3} = 0. \qquad (7.2)$$

Using (5.8), we write (7.2) as

$$\tfrac{1}{2}T_{A_1P'A_2}{}^{P'}{}_{QQ'}\tilde{\nabla}^{QQ'}\phi^{A_1A_2}{}_{A_3}+\tfrac{1}{4}\tilde{R}_{QP'A_1}{}^{P'A_1}{}_{Q'A_2}{}^{Q'}\phi^{QA_2}{}_{A_3}+\tfrac{1}{4}\tilde{R}_{QP'A_2}{}^{P'A_2}{}_{Q'A_1}{}^{Q'}\phi^{QA_1}{}_{A_3}$$
$$-\tfrac{1}{4}\tilde{R}_{A_3P'Q}{}^{P'}{}_{A_1Q'A_2}{}^{Q'}\phi^{A_1A_2Q}=0, \tag{7.3}$$

or, after using the spinor decompositions (3.8) and (3.9), as

$$\tfrac{1}{2}T_{A_1P'A_2}{}^{P'}{}_{QQ'}\tilde{\nabla}^{QQ'}\phi^{A_1A_2}{}_{A_3}+10\tilde{\Theta}_{A_1A_2}\phi^{A_1A_2}{}_{A_3}-\tilde{\Psi}_{A_3A_1A_2Q}\phi^{A_1A_2Q}=0. \tag{7.4}$$

In the limit as $T_{\alpha\beta}{}^{\gamma}\to 0$ we have

$$\psi_{A_3A_1A_2Q}\phi^{A_1A_2Q}=0. \tag{7.5}$$

This is the same $(\frac{3}{2},0)$ consistency condition obtained in Christensen and Duff (1979). In that paper we concluded that if $\phi^{A_1A_2Q}\neq 0$, then we must have $\psi_{A_3A_1A_2Q}=0$, i.e. the Weyl curvature must be anti-self-dual. Obviously, this simple conclusion no longer holds for the more complicated condition (7.4).

The reader will note that we have not decomposed the first term in (7.4). As we shall see, the properties of the $T_{A_1P'A_2}{}^{P'}{}_{QQ'}$ term are more easily discussed without immediate decomposition. Consider the $\varepsilon_{\alpha\beta\gamma\delta}$ symbol discussed briefly in § 2. It is straightforward to show that

$$\varepsilon_{\alpha\beta\gamma\delta}=\varepsilon_{AA'BB'CC'DD'}=\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'}-\varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'C'}\varepsilon_{B'D'} \tag{7.6}$$

is the spinor decomposition of $\varepsilon_{\alpha\beta\gamma\delta}$. Forming

$${}^*T_{\alpha\beta}{}^{\rho}\equiv\tfrac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu}T_{\mu\nu}{}^{\rho}$$

and using (7.6) we find

$${}^*T_{AA'BB'}{}^{PP'}=-T_{AB'BA'}{}^{PP'}. \tag{7.7}$$

From definition (2.29) come the self-dual and anti-self-dual parts of $T_{\alpha\beta}{}^{\gamma}$

$${}^+T_{AA'BB'}{}^{PP'}=\tfrac{1}{2}(T_{AA'BB'}{}^{PP'}-T_{AB'BA'}{}^{PP'})=\tfrac{1}{2}\varepsilon_{A'B'}T_{AQ'B}{}^{Q'PP'},$$
$${}^-T_{AA'BB'}{}^{PP'}=\tfrac{1}{2}(T_{AA'BB'}{}^{PP'}+T_{AB'BA'}{}^{PP'})=\tfrac{1}{2}\varepsilon_{AB}T_{QA'}{}^{Q}{}_{B'}{}^{PP'}.$$

Using these we can write

$$T_{AP'B}{}^{P'}{}_{QQ'}={}^+T_{AP'B}{}^{P'}{}_{QQ'}\equiv{}^+T_{ABQQ'},$$
$$T_{PA'}{}^{P}{}_{B'QQ'}={}^-T_{PA'}{}^{P}{}_{B'QQ'}\equiv{}^-T_{A'B'Q'Q}. \tag{7.8}$$

From (3.3) come the decompositions

$${}^+T_{ABCC'}=T_{ABCC'}+\varepsilon_{BC}U_{AC'}+\varepsilon_{AC}U_{BC'},$$
$${}^-T_{A'B'C'C}=\bar{T}_{A'B'C'C}+\varepsilon_{B'C'}\bar{U}_{A'C}+\varepsilon_{A'C'}\bar{U}_{B'C}. \tag{7.9}$$

Looking at (7.4) again, we see that the three terms are all constructed from the self-dual parts of tensors, that is, the torsion tensor, the antisymmetric part of the Ricci tensor (see (3.18)) and the Weyl tensor $\tilde{C}_{\alpha\beta\gamma\delta}$.

For a field transforming like $(A,0)$ with $A=\frac{3}{2},\frac{5}{2},\ldots$, the consistency conditions are

$$\tfrac{1}{2}{}^+T_{A_1A_2QQ'}\tilde{\nabla}^{QQ'}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}}+4(A+1)\tilde{\Theta}_{A_1A_2}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}}$$
$$-\sum_{i=3}^{2A}\tilde{\Psi}_{A_iA_1A_2Q}\phi^{A_1A_2}{}_{A_3\ldots}{}^{Q}{}_{\ldots A_{2A}}=0. \tag{7.10}$$

What do the conditions (7.10) tell us? If we require that $\phi_{A_1\ldots A_{2A}} \neq 0$ and $\tilde{\nabla}^{QQ'}\phi_{A_1\ldots A_{2A}} \neq 0$, then we can satisfy (7.10) by choosing

$$ {}^{+}T_{A_1A_2QQ'} = 0, \qquad \tilde{\Theta}_{A_1A_2} = 0, \qquad \tilde{\Psi}_{A_iA_1A_2Q} = 0, \tag{7.11}$$

i.e. by taking the torsion, antisymmetric part of the Ricci tensor and the Weyl tensor to be anti-self-dual. Under these conditions, if we require that fields $(A, 0)$ and $(0, A)$ both appear, we force $T_{\alpha\beta}{}^{\gamma} = 0$ and we are led back to the old consistency conditions. This is a drastic step and defeats the purpose of adding torsion in the first place. All that we can really say is that given a manifold with curvature and torsion, the field $\phi_{A_1\ldots A_{2A}}$ and its first covariant derivative must satisfy relation (7.10). Alternatively, given a field $\phi_{A_1\ldots A_{2A}}$ and its covariant derivative, the torsion and curvature must have values that obey (7.10).

For fields transforming according to a representation other than $(A, 0)$ or $(0, A)$, we get slightly different conditions. Look at the spin-$\frac{3}{2}$ field $(1, \frac{1}{2})$. The consistency conditions are

$$\tilde{\nabla}_{H'(A_1}\tilde{\nabla}_{A_2)}{}^{H'}\phi^{A_1A_2}{}_{A_1'} = 0 \tag{7.12}$$

or

$$\tfrac{1}{2}T_{A_1P'A_2}{}^{P'}{}_{QQ'}\tilde{\nabla}^{QQ'}\phi^{A_1A_2}{}_{A_1'} + \tfrac{1}{4}\tilde{R}_{QP'A_1}{}^{P'A_1}{}_{Q'A_2}{}^{Q'}\phi^{QA_2}{}_{A_1'}$$
$$+\tfrac{1}{4}\tilde{R}_{QP'A_2}{}^{P'A_2}{}_{Q'A_1}{}^{Q'}\phi^{A_1Q}{}_{A_1'} - \tfrac{1}{4}\tilde{R}_{PA_1'}{}^{P}{}_{Q'A_1P'A_2}{}^{P'}\phi^{A_1A_2Q'} = 0. \tag{7.13}$$

The spinor decomposition of this condition is

$$\tfrac{1}{2}{}^{+}T_{A_1A_2QQ'}\tilde{\nabla}^{QQ'}\phi^{A_1A_2}{}_{A_1'} + 8\tilde{\Theta}_{A_1A_2}\phi^{A_1A_2}{}_{A_1'} - \tfrac{1}{2}\tilde{\bar{\Phi}}_{A_1'Q'A_1A_2}\phi^{A_1A_2Q'} = 0. \tag{7.14}$$

For fields transforming like $(A, \frac{1}{2})$ with $A = 1, 2, 3, \ldots$, (7.14) generalises to

$$\tfrac{1}{2}{}^{+}T_{A_1A_2QQ'}\tilde{\nabla}^{QQ'}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}A_1'} + 4(A+1)\tilde{\Theta}_{A_1A_2}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}A_1'}$$
$$-\sum_{i=3}^{2A}\tilde{\Psi}_{A_iA_1A_2Q}\phi^{A_1A_2}{}_{A_3\ldots}{}^{Q}{}_{\ldots A_{2A}A_1'} - \tfrac{1}{2}\tilde{\bar{\Phi}}_{A_1'Q'A_1A_2}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}}{}^{Q'} = 0. \tag{7.15}$$

It is now obvious that the consistency conditions for fermion fields transforming according to the (A, B) representation with $A > B$ are

$$\tfrac{1}{2}{}^{+}T_{A_1A_2QQ'}\tilde{\nabla}^{QQ'}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}A_1'\ldots A_{2B}'} + 4(A+1)\tilde{\Theta}_{A_1A_2}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}A_1'\ldots A_{2B}'}$$
$$-\sum_{i=3}^{2A}\tilde{\Psi}_{A_iA_1A_2Q}\phi^{A_1A_2}{}_{A_3\ldots}{}^{Q}{}_{\ldots A_{2A}A_1'\ldots A_{2B}'}$$
$$-\tfrac{1}{2}\sum_{i=1}^{2B}\tilde{\bar{\Phi}}_{A_i'Q'A_1A_2}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}A_1'\ldots}{}^{Q'}{}_{\ldots A_{2B}'} = 0. \tag{7.16}$$

Following a similar path, we get the consistency conditions for fermion fields transforming according to the (A, B) representation with $A < B$. They are

$$\tfrac{1}{2}{}^{-}T_{A_1'A_2'Q'Q}\tilde{\nabla}^{QQ'}\phi_{A_1\ldots A_{2A}}{}^{A_1'A_2'}{}_{A_3'\ldots A_{2B}'} + 4(B+1)\tilde{\bar{\Theta}}_{A_1'A_2'}\phi_{A_1\ldots A_{2A}}{}^{A_1'A_2'}{}_{A_3'\ldots A_{2B}'}$$
$$-\sum_{i=3}^{2B}\tilde{\bar{\Psi}}_{A_i'A_1'A_2'Q'}\phi_{A_1\ldots A_{2A}}{}^{A_1'A_2'}{}_{A_3'\ldots}{}^{Q'}{}_{\ldots A_{2B}'}$$
$$-\tfrac{1}{2}\sum_{i=1}^{2A}\tilde{\Phi}_{A_iQA_1'A_2'}\phi_{A_1\ldots}{}^{Q}{}_{\ldots A_{2A}}{}^{A_1'A_2'}{}_{A_3'\ldots A_{2B}'} = 0. \tag{7.17}$$

The boson field equations we have chosen are

$$-\tilde{\Box}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}+2A\tilde{\nabla}_{H'(A_1}\tilde{\nabla}_{H)}{}^{H'}\phi^{H}{}_{A_2\ldots A_{2A}A'_1\ldots A'_{2B}}$$
$$+2B\tilde{\nabla}_{H(A'_1}\tilde{\nabla}_{H')}{}^{H}\phi_{A_1\ldots A_{2A}}{}^{H'}{}_{A'_2\ldots A'_{2B}}=0. \qquad (7.18)$$

Hitting this with $\varepsilon^{A_1A_2}$ gives the condition

$$2A\tilde{\nabla}_{H'(A_1}\tilde{\nabla}_{A_2)}{}^{H'}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}A'_1\ldots A'_{2B}}=0, \qquad (7.19)$$

while hitting (7.18) with $\varepsilon^{A'_1A'_2}$ gives

$$2B\tilde{\nabla}_{H(A'_1}\tilde{\nabla}_{A'_2)}{}^{H}\phi_{A_1\ldots A_{2A}}{}^{A'_1A'_2}{}_{A'_3\ldots A'_{2B}}=0. \qquad (7.20)$$

It is easy to see that the $2A$ and $2B$ factors in (7.19) and (7.20) are superfluous. Consider (7.19) for example. In order to form the condition in the first place we must have $A\geqslant 1$. Obviously $2A\geqslant 2$ and so we can simply divide out the $2A$ factor. Similarly the $2B$ factor drops off from (7.20). Hence, the boson consistency conditions become

$$\tilde{\nabla}_{H'(A_1}\tilde{\nabla}_{A_2)}{}^{H'}\phi^{A_1A_2}{}_{A_3\ldots A_{2A}A'_1\ldots A'_{2B}}=0,$$
$$\tilde{\nabla}_{H(A'_1}\tilde{\nabla}_{A'_2)}{}^{H}\phi_{A_1\ldots A_{2A}}{}^{A'_1A'_2}{}_{A'_3\ldots A'_{2B}}=0, \qquad (7.21)$$

which are identical in form to the fermion ones. Bosons must satisfy both (7.16) *and* (7.17).

Suppose we are given non-zero $\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}$ and $\tilde{\nabla}^{QQ'}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}$. The consistency conditions divide up the possible manifolds into classes defined by which components of the torsion and curvature tensors are restricted:

Manifold type	Components restricted	Manifold type	Components restricted
I	None	V	$^+T, \tilde{\Theta}, \tilde{\Psi}, \bar{\bar{\tilde{\Phi}}}$
II	$^+T, \tilde{\Theta}$	$\overline{\mathrm{V}}$	$^-T, \bar{\bar{\tilde{\Theta}}}, \bar{\bar{\tilde{\Psi}}}, \tilde{\Phi}$
$\overline{\mathrm{II}}$	$^-T, \bar{\bar{\tilde{\Theta}}}$	VI	$^+T, ^-T, \tilde{\Theta}, \bar{\bar{\tilde{\Theta}}}, \Phi, \bar{\bar{\tilde{\Phi}}}$
III	$^+T, \tilde{\Theta}, \Psi$	VII	$^+T, ^-T, \tilde{\Theta}, \bar{\bar{\tilde{\Theta}}}, \tilde{\Phi}, \bar{\bar{\tilde{\Phi}}}, \tilde{\Psi}$
$\overline{\mathrm{III}}$	$^-T, \bar{\bar{\tilde{\Theta}}}, \bar{\bar{\tilde{\Psi}}}$	$\overline{\mathrm{VII}}$	$^+T, ^-T, \tilde{\Theta}, \bar{\bar{\tilde{\Theta}}}, \tilde{\Phi}, \bar{\bar{\tilde{\Phi}}}, \bar{\bar{\tilde{\Psi}}}$
IV	$^+T, \tilde{\Theta}, \bar{\bar{\tilde{\Phi}}}$	VIII	$^+T, ^-T, \tilde{\Theta}, \bar{\bar{\tilde{\Theta}}}, \tilde{\Phi}, \bar{\bar{\tilde{\Phi}}}, \tilde{\Psi}, \bar{\bar{\tilde{\Psi}}}$
$\overline{\mathrm{IV}}$	$^-T, \bar{\bar{\tilde{\Theta}}}, \tilde{\Phi}$		

The results of the consistency conditions for fields transforming according to the (A, B) representation of SO(4) with $A+B\leqslant 3$ are summarised in the following table.

B \ A	0	1/2	1	3/2	2	5/2	3
0	I	I	II	III	III	III	III
1/2	I	I	IV	V	V	V	V
1	$\overline{\mathrm{II}}$	$\overline{\mathrm{IV}}$	VI	V	VII	V	VII
3/2	$\overline{\mathrm{III}}$	$\overline{\mathrm{V}}$	$\overline{\mathrm{V}}$	VIII	V	VIII	V
2	$\overline{\mathrm{III}}$	$\overline{\mathrm{V}}$	$\overline{\mathrm{VII}}$	$\overline{\mathrm{V}}$	VIII	V	VIII
5/2	$\overline{\mathrm{III}}$	$\overline{\mathrm{V}}$	$\overline{\mathrm{V}}$	VIII	$\overline{\mathrm{V}}$	VIII	V
3	$\overline{\mathrm{III}}$	$\overline{\mathrm{V}}$	$\overline{\mathrm{VII}}$	$\overline{\mathrm{V}}$	VIII	$\overline{\mathrm{V}}$	VIII

For example, a non-zero field $\phi_{A_1A_2A_3A_4A'_1A'_2}$ $[\phi(2, 1)]$ forces restrictions of types VII.

It is important to note that if we ask that, for example, both $(1, \frac{1}{2})$ *and* $(\frac{1}{2}, 1)$ type fields exist, then the restrictions are type VI, the union of IV and $\overline{\text{IV}}$. It is also crucial to remember that these restrictions *do not* necessarily force any component of the torsion or curvature to be zero as they can do in the torsion-free case (see Christensen and Duff 1979).

We can now discuss our reasons for choosing (7.18) as our field equations rather than those we can obtain from (6.46)

$$-\tilde{\Box}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}+\sum_{i=1}^{2A}\tilde{\nabla}_{H'(A_i}\tilde{\nabla}_{H)}{}^{H'}\phi_{A_1\ldots}{}^{H}{}_{\ldots A_{2A}A'_1\ldots A'_{2B}}$$

$$+\sum_{i=1}^{2B}\tilde{\nabla}_{H(A'_i}\tilde{\nabla}_{H')}{}^{H}\phi_{A_1\ldots A_{2A}A'_1\ldots}{}^{H'}{}_{\ldots A'_{2B}}=0. \qquad (7.22)$$

If we contract $\varepsilon^{A_1A_2}$ or $\varepsilon^{A'_1A'_2}$ into (7.22) we get $0=0$. These field equations are already consistent! It seems that (7.22) might be a better choice. We would not have to contend with complicated boson consistency conditions. However, there are several reasons, all related, to prefer the equations (7.18).

In previous work on the torsion-free case (Christensen and Duff 1979), we studied index theorems and found that forming them required the commutativity property

$$\nabla_{A'}{}^{A_1}\Delta\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}}=\Delta\nabla_{A'}{}^{A_1}\phi_{A_1\ldots A_{2A}A'_1\ldots A'_{2B}} \qquad (7.23)$$

which in turn was satisfied only when the boson consistency conditions hold. The derivation of index theorems in the torsion case will require the same sort of condition. Also, if we want the torsion case to reduce to the torsion-free case, then we must have the torsion consistency conditions reduce to the torsionless ones when $T_{\alpha\beta}{}^{\gamma}=0$. This would not happen with (7.22), but does with (7.18).

In the torsion-free case we also studied the idea of a super index theorem. These theorems relate boson and fermion zero modes. As is standard in supergravity theories, bosons and fermions are put into multiplets. It seems unlikely that one set of fields in a multiplet would satisfy consistency conditions while another set does not. In any case the existence of fermions on a curved manifold with torsion will require some conditions on the components of the torsion and curvature.

Finally, the consistency conditions can give us clues for building a theory which is consistent from the start. For example, suppose that we know nothing about supergravity. From the table we see that the (1, 1) portion of the spin-2 field (the graviton) forces conditions of type VI while the spin-$\frac{3}{2}$ $(1, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ fields give the same conditions. One could guess that a coupling of the spin-2 and spin-$\frac{3}{2}$ fields through a theory with torsion might give a consistent theory. This is exactly what supergravity does. The torsion is directly related to the spin-$\frac{3}{2}$ field and any possible inconsistent terms are eliminated.

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Note added. After the manuscript of this paper was completed we obtained a copy of a 1981 Moscow University Report 'Spectral Geometry of the Riemann–Cartan Space–Time' by Yu N Obukhov. Higher spin field equations and their consistency conditions are also discussed in this report. Obukhov's results are different from ours. In particular, his consistency conditions appear to be more restrictive.

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